Orthogonal decomposition of derivatives and antiderivatives for easy evaluation of extended Gram matrix

M. Telescu, P. Bréhonnet, N. Tanguy, P. Vilbé, and L.C. Calvez

Laboratoire d'Electronique et Systèmes de Télécommunications (LEST),
UMR CNRS 6165 - site Université de Bretagne Occidentale (UBO),
6, av. Le Gorgeu, CS 93837, 29238 BREST Cedex 3, France

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Abstract

Simple and efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of a function with rational Laplace transform are proposed. Based on a new theorem related to Routh $\alpha-\beta$ expansion, they enable direct evaluation of the extended Gram matrix which has proved to be very useful in model-reduction applications.

1 - Introduction

Since the pioneering work of Jain [1, 2] the Gram matrix has proved to be one of the most reliable tools in the field of model reduction [3-6]; its ability to produce good pole positions has motivated a number of papers. Early work has been devoted to the Gram matrix of successive antiderivatives of a time response. Some of the authors [3] have shown that evaluation of this Gram matrix can be achieved in the frequency domain; in a sequel to [3], Lucas [7] and Sreeram and Goddard [8] presented additional simplifications in computation. Sreeram and Agathoklis [5,9] have shown the connection between the Gram matrix and the other Gramians, whereby the Gram matrix can be obtained by solving Lyapunov equations.

In [4,6] the Gram matrix has been successfully extended to include both derivatives and antiderivatives. It is the purpose of this communication to propose a straightforward orthogonal decomposition of these derivatives and antiderivatives, directly obtained from the so-called Routh $\alpha-\beta$ parameters. Owing to this orthogonal representation, an elementary and simple computation of the extended Gram matrix follows readily

2 – Background

Although the Routh $\alpha-\beta$ tables are familiar to model reduction researchers, to make this paper self contained and to state precisely some notations and numberings, the procedure is first outlined below in polynomial form; for a tabular form see [10].

Let $D(s)=\sum_{i=0}^{n} d_i s^i$ denote a strictly Hurwitz polynomial. Starting with its even and odd parts, $D_n(s)=d_n s^n + d_{n-2}s^{n-2}+\ldots$ and $D_{n-1}(s)=d_{n-1}s^{n-1}+d_{n-3}s^{n-3}+\ldots$, let a sequence of polynomials of descending degree be computed recursively by the formula

$$D_{k-1} = D_{k+1} - \alpha_k s D_k, \quad k = n-1,\ldots,1 \tag{1}$$

with $\alpha_k = lc(D_{k+1})/lc(D_k), \quad k = n-1,\ldots,0$, where $lc(P)$ denotes the leading coefficient of the polynomial P. Notice that the numbering as defined in [10] has been modified for convenience. A strictly proper rational Laplace transform $F(s)=N(s)/D(s)$ that is asymptotically stable can always be decomposed into the following form

$$F(s)=\sum_{k=0}^{n-1}\beta_k D_k(s)/D(s) \tag{2}$$

in which the coefficients β_k are uniquely determined by $N(s)=\sum_{k=0}^{n-1}\beta_k D_k(s)$. It is known [11] that the Laplace transforms $\Phi_k(s)\hat{=} D_k(s)/D(s)$, $k=0,...,n-1$, define a set of n orthogonal time functions $\varphi_k(t)$:

$$\langle\varphi_k,\varphi_i\rangle \hat{=} \int_0^\infty \varphi_k(t)\varphi_i(t)dt=\delta_{ki}/(2\alpha_k) \tag{3}$$

As far as the authors are aware, the following theorem, which is the keystone of the two algorithms to be proposed in this paper, has never been mentioned before.

Theorem 1: Define $\varphi_{-1}\hat{=}0$ and $\varphi_n \hat{=} -\varphi_{n-1}$. Then the derivative of $\varphi_k(t)$ satisfies

$$\alpha_k \frac{d\varphi_k(t)}{dt}=\varphi_{k+1}(t)-\varphi_{k-1}(t),\ k=0,...,n-1 \tag{4}$$

Proof of Theorem 1: On the understanding that $D_{-1}=0$, eqn. 1 can be rewritten as

$$sD_k=(D_{k+1}-D_{k-1})/\alpha_k,\ k=0,...,n-1 \tag{5}$$

Assume $k\le n-2$. The initial value theorem for the Laplace transform yields $\varphi_k(+0)=0$, therefore the Laplace transform of $d\varphi_k(t)/dt$ is sD_k/D. Thus, starting from eqn. 5 and dividing throughout by D, eqn. 4 is proved for $k=0,...,n-2$. Now, consider $\varphi_{n-1}(+0)=d_{n-1}/d_n=1/\alpha_{n-1}$, hence the Laplace transform of $d\varphi_{n-1}(t)/dt$ is $(sD_{n-1}/D_n)-1/\alpha_{n-1}$ which, on account of eqn. 5 and $D_n+D_{n-1}=D$ may be written $-(D_{n-1}+D_{n-2})/(\alpha_{n-1}D)$. This achieves the proof of eqn. 4 for $k=n-1$.

An extended Gram matrix involves inner products of signals $f_i(t)$, $i\in Z$, recursively defined by $f_0(t)\hat{=} f(t)$, $f_{i+1}(t)\hat{=} df_i(t)/dt$ and $f_{i-1}(t)\hat{=}\int_\infty^t f_i(\tau)d\tau$. Applying the inverse Laplace transform to eqn. 2 yields the orthogonal decomposition $f_0(t)=\sum_{k=0}^{n-1}\beta_k\varphi_k(t)$. More generally, owing to the pole preserving property of operators d/dt and $\int_\infty^t$, any $f_i(t)$ admits an orthogonal decomposition of the following form

$$f_i(t)=\sum_{k=0}^{n-1}\beta_k^i\varphi_k(t) \tag{6}$$

Starting from $\beta_k^0\hat{=}\beta_k$, the coefficients β_k^i can be efficiently computed for $i=1,2,...$ and $i=-1,-2,...$ by algorithms D^+ and D^- proposed below.

Algorithm D^+: Given α_k, β_k^i and $\theta_k^i \hat{=} \beta_k^i/\alpha_k$ relative to $f_i(t)$, let $\theta_{-1}^i\hat{=}0$ and $\theta_n^i\hat{=}\theta_{n-1}^i$; then the following algorithm computes the β's and θ's relative to the derivative $f_{i+1}(t)$:

For $k=0,...,n-1$

$$\beta_k^{i+1}:=\theta_{k-1}^i-\theta_{k+1}^i$$
$$\theta_k^{i+1}:=\beta_k^{i+1}/\alpha_k$$

Proof of Algorithm D^+: Differentiating eqn. 6 with respect to t, using eqn. 4 and rearranging yields $f_{i+1}(t)=\sum_{k=0}^{n-1}(\theta_{k-1}^i-\theta_{k+1}^i)\varphi_k(t)$ which achieves the proof. This algorithm takes only $(n-1)$ additive operations $(+,-)$ and n multiplicative operations $(\times,\div)$. For comparison, the standard way to compute β_k^{i+1} is to evaluate the Laplace transform of $f_{i+1}(t)$ from that of $f_i(t)$ and then to construct the related $\beta-$table. The first step takes exactly the same number of operations as algorithm D^+. Hence, the orthogonal decomposition of $f_{i+1}(t)$ via algorithm D^+ can be obtained as cheaply as its Laplace transform.

In short, algorithm D^+ saves the $\left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$ multiplicative operations and $\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right)$ additive operations required by the $\beta-$table, where $\lfloor x \rfloor$ denotes the integer part of x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Algorithm D^-: Given α_k, β_k^i and $\theta_k^i \mathrel{\hat{=}} \beta_k^i / \alpha_k$ relative to $f_i(t)$, define $\theta_{-1}^{i-1} \mathrel{\hat{=}} 0$; then the following algorithm computes β's and θ's relative to the antiderivative $f_{i-1}(t)$:

$$
\begin{aligned}
&\text{For } k = 0,\ldots,\left\lfloor \frac{n-2}{2} \right\rfloor \\
&\qquad \theta_{2k+1}^{i-1} := \theta_{2k-1}^{i-1} - \beta_{2k}^i \\
&\qquad \text{if } n \text{ is odd then } \theta_{n-1}^{i-1} := \theta_{n-2}^{i-1} - \beta_{n-1}^i \\
&\qquad\qquad\qquad\quad \text{else } \theta_{n-2}^{i-1} := \theta_{n-1}^{i-1} + \beta_{n-1}^i \\
&\text{For } k = \left\lfloor \frac{n-1}{2} \right\rfloor,\ldots,1 \\
&\qquad \theta_{2k-2}^{i-1} := \theta_{2k}^{i-1} + \beta_{2k-1}^i \\
&\text{For } k = 0,\ldots,n-1 \\
&\qquad \beta_k^{i-1} := \alpha_k \theta_k^{i-1}
\end{aligned}
$$

Proof of Algorithm D^-: Replacing i by $i-1$ in algorithm D^+ we have $\beta_k^i = \theta_{k-1}^{i-1} - \theta_{k+1}^{i-1}$, $\theta_k^i = \beta_k^i / \alpha_k$. Letting $k = 0$, with $\theta_{-1}^{i-1} = 0$ in mind, yields $\theta_1^{i-1} = -\beta_0^i$; then it is a simple matter to see that the θ_k^{i-1} can be computed in succession, in the order indicated by algorithm D^-. Once again the orthogonal decomposition is obtained as cheaply as the Laplace transform and all the operations required by the $\beta-$table are saved.

Theorem 2: Let $B = [b_{ij}]$ and $\Theta = [\theta_{ij}]$ denote $(m+p+1)\times n$ matrices with (i, j) entries respectively given by $b_{ij} = \beta_{j-1}^{i-p-1}$ and $\theta_{ij} = \theta_{j-1}^{i-p-1}$. Then, denoting transposition by T, the extended Gram matrix involving m antiderivatives and p derivatives is given by

$$
G(f_{-m},\ldots, f_0,\ldots, f_p) = (1/2)B\Theta^T \tag{7}
$$

Proof: Using eqn. 6 and the orthogonality property of eqn. 3, any entry $\langle f_i, f_j \rangle$ of G is readily written as $\langle f_i, f_j \rangle = \sum_{k=0}^{n-1} \beta_k^i \beta_k^j / (2\alpha_k) = \sum_{k=0}^{n-1} \beta_k^i \theta_k^j / 2$ by which eqn. 7 follows

3 – Illustrative examples

We first consider the transfer function given by Krajewski *et al.* in [6]

$$
F(s) = \frac{s^2 + 10s + 100}{1.21s^4 + 3s^3 + 110s^2 + 230s + 100}
$$

with a view to deriving $G(f_{-1}, f_0, f_1)$ and a second-order reduced model. The entries in rows 2 of B and Θ are readily obtained by the standard $\alpha-\beta$ Routh algorithm. A run of algorithm D^- yields the entries in rows 1 and a run of algorithm D^+ yields the entries in rows 3:

$$G = \frac{1}{2}\begin{bmatrix} -2.026 & -0.076 & -0.174 & -0.403 \\ 0.942 & 0.047 & 0.058 & 0.000 \\ -0.580 & 0.110 & 0.580 & 0.333 \end{bmatrix}\begin{bmatrix} -0.953 & -0.942 & -1.000 & -1.000 \\ 0.443 & 0.580 & 0.333 & 0.000 \\ -0.273 & 1.354 & 3.333 & 0.826 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1.290 & -0.5 & -0.232 \\ -0.5 & 0.232 & 0 \\ -0.232 & 0 & 1.258 \end{bmatrix}$$

After the matrix product has been computed, the method described in [4] yields the $(1,-1,2)$ approximant whose second-order denominator matches exactly that of eqn. 44 in [6]. The squared L_2 norm of the error is equal to 1.125×10^{-2} to be compared with 1.95×10^{-2} obtained through balancing as pointed out in [6]. Note that the L_2 optimal value obtained via a Gauss-Newton nonlinear optimisation procedure is 1.099×10^{-2} .

In the case of the transfer function given by Hwang and Chen in [13],

$$F(s) = \frac{9s^3 + 42s^2 + 31s + 10}{s^4 + 8s^3 + 21s^2 + 22s + 8}$$

with a view to deriving $G(f_0, f_1, f_2)$ given by

$$G = \begin{bmatrix} 11.40 & -40.50 & 111.7 \\ -40.50 & 158.2 & -450.0 \\ 111.7 & -450.0 & 0.451 \end{bmatrix}$$

and a second order model as in the previous example, the technique described in [4] yields the (0,1,2) approximant with an error norm of 1.5142×10^{-2} while the approximant given by [13] yields 8.2904×10^{-2}.
The optimal L_2 value in this case, calculated by Lucas in [14], is 7.0135×10^{-3}.

We finally refer to Pal's celebrated example [12]:

$$F\ (s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}$$

with the corresponding Gram matrix G:

$$G = \begin{bmatrix} 0.451 & -0.125 & -1.194 \\ -0.125 & 0.694 & -0.5 \\ -1.194 & -0.5 & 9.222 \end{bmatrix}$$

This time we use Jain's model order reduction method [2] as described in [3] which yields an error norm of 3.007×10^{-2} to be compared to Pal's original model yielding 4.098×10^{-2}. We may observe that our result is in fact very close to the optimal value obtained via Gauss-Newton optimisation providing 2.909×10^{-2}.

4 – Conclusion

Efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of functions with rational Laplace transforms have been presented. A simple method for computing the extended Gram matrix follows, whereby model order reduction very close to the optimal can be carried out without any optimising iterative procedure

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